SEPARATING PRIME BLOCKS AND PHASE-RESOLVED ASYMPTOTICS FOR HARMONIC SUBSET SUMS

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Abstract. For a finite set $X \subset \mathbb{N}$, let

$$E(X) = \left\{ \sum_{n \in A} \frac{1}{n} : A \subseteq X \right\}, \qquad S(N) = |E([N])| .$$

A recent prime-block argument of Luo, Yang, and Zhu gives the order of magnitude

$$\log S(N) \asymp \frac{N}{\log N} \prod_{j=3}^{\kappa(N)} \log_j N,$$

where $\kappa(N)$ is the last iterated logarithm above a fixed threshold. The argument is one-sided at the block level and uses a running maximum, so it does not itself produce a ratio asymptotic.

We introduce a modular separation condition for a large prime block. For every separating prime the corresponding block is an exact direct summand at the level of reciprocal subset sums. A discriminant argument, together with the known upper bound, shows that all but $N^{o(1)}$ of the relevant large primes are separating. With

$$F(N) := \frac{\log N}{N} \log S(N),$$

this upgrades the prime-block inequality to the asymptotic renewal identity

$$F(N) = \frac{5 \log 2}{6} + \sum_{3 \le m \le \log N} \frac{F(m)}{(m+1) \log m} + o(1).$$

After a double-logarithmic change of variables, the renewal identity implies slow variation and

$$F(N) \sim \log_3 N \, F(\log N).$$

We then identify the unavoidable terminal oscillation. For every fixed $A > 1$ there is a positive phase function h_A on $[A, \mathrm{e}^A]$ such that

$$F(N) = L_A(N)\big(h_A(u_A(N)) + o(1)\big),$$

where L_A is the hard-cutoff iterated-logarithmic product and $u_A(N)$ is the final iterate in $(A, \mathrm{e}^A]$. The phase is given by a uniformly convergent tower limit and satisfies the exact cocycle relation

$$h(\mathrm{e}^u) = u h(u).$$

Consequently no positive constant C can satisfy $F(N) \sim C L_A(N)$ for a hard cutoff $A > 1$. The phase is rigorously identified, but a closed form, effective high-precision evaluation, and regularity beyond boundedness remain open.

Contents

1. Introduction and statement of results

For a finite set $X \subset \mathbb{N}$, write

$$E(X) := \left\{ \sum_{n \in A} \frac{1}{n} : A \subseteq X \right\}, \qquad S(N) := |E([N])|, \qquad s(N) := \log S(N). \tag{1.1}$$

All logarithms are natural. We use

$$\log_0 x = x, \qquad \log_{j+1} x = \log(\log_j x)$$

whenever the expression is defined. For a fixed $A > 1$, set

$$\kappa_A(N) := \max(\{j \geq 3 : \log_j N > A\} \cup \{2\}), \qquad L_A(N) := \prod_{j=3}^{\kappa_A(N)} \log_j N, \tag{1.2}$$

with the empty product equal to 1.

Bleicher and Erdős initiated the study of $S(N)$, and Bettin, Grenié, Molteni, and Sanna proved a lower bound reaching the natural terminal iterated-logarithmic scale. Building on these works, Luo, Yang, and Zhu obtained

$$\log S(N) \asymp_A \frac{N}{\log N} L_A(N) \tag{1.3}$$

for every fixed $A > 1$. Their upper bound starts from disjoint large-prime blocks and then replaces the resulting weighted sum by a running maximum. The present note retains the arithmetic information discarded in that replacement.

The first result is purely algebraic. Put

$$D_m := \operatorname{lcm}(1, \dots, m), \qquad \mathcal{A}_m := D_m E([m]) \subset \mathbb{Z}. \tag{1.4}$$

Thus $|\mathcal{A}_m| = S(m)$.

Definition 1.1. Let $p > m$ be prime. We say that p is *separating for* m if reduction modulo p is injective on $\mathcal{A}_m$.

Theorem 1.2 (Exact separation of good prime blocks)**.** *Let $N, M \in \mathbb{N}$ satisfy*

$$M < q := \frac{N}{M+1}, \qquad q^2 > N.$$

For each prime $p > q$, put $m_p = \lfloor N/p \rfloor$ and

$$B_p = \{p, 2p, \dots, m_p p\}.$$

Let $\mathcal{G}_N$ be the primes $p > q$ that are separating for m_p, and set

$$Z_N = [N] \setminus \bigcup_{p \in \mathcal{G}_N} B_p.$$

Then the addition map

$$E(Z_N) \times \prod_{p \in \mathcal{G}_N} \frac{1}{p} E([m_p]) \longrightarrow E([N])$$

is bijective. In particular,

$$S(N) = |E(Z_N)| \prod_{p \in \mathcal{G}_N} S(m_p). \tag{1.5}$$

The separating condition fails only for primes dividing a finite discriminant. Combining this observation with the upper half of (1.3) makes the exceptional set negligible.

Theorem 1.3 (Asymptotic renewal identity)**.** *Assume the upper bound in (1.3), which follows from the prime-block argument of Luo–Yang–Zhu with the minor correction recorded in Section 3. Then*

$$F(N) = \frac{5 \log 2}{6} + \sum_{3 \le m \le \log N} \frac{F(m)}{(m+1) \log m} + o(1), \qquad F(N) := \frac{\log N}{N} s(N). \tag{1.6}$$

The error is two-sided. In particular, no separate lower-bound construction is needed to pass from the prime blocks to (1.6).

The known lower bound is used next only to control the size of F from below and hence to identify the renewal profile.

Theorem 1.4 (One-step ratio asymptotic)**.** *Assume the two-sided estimate (1.3). Then*

$$F(N) \sim \log_3 N \, F(\log N). \tag{1.7}$$

More generally, for every fixed $r \ge 1$,

$$F(N) \sim \left(\prod_{j=3}^{r+2} \log_j N \right) F(\log_r N). \tag{1.8}$$

The iteration cannot be terminated by a single constant because a hard cutoff has deterministic jumps. The correct terminal object is a phase function.

Theorem 1.5 (Phase-resolved asymptotic)**.** *Assume* (1.3)*. For every fixed* $A > 1$*, define*

$$u_A(N) := \log_{\kappa_A(N)} N \in (A, \mathrm{e}^A]. \tag{1.9}$$

There exists a positive bounded function $h_A : [A, \mathrm{e}^A] \to (0, \infty)$ *such that, uniformly as* $N \to \infty$*,*

$$F(N) = L_A(N)\big(h_A(u_A(N)) + o(1)\big). \tag{1.10}$$

The functions h_A *are restrictions of a single positive function* $h : (1, \infty) \to (0, \infty)$ *satisfying*

$$h(\mathrm{e}^u) = uh(u). \tag{1.11}$$

They admit the tower-limit representation in (8.5) *and the infinite-product representation in* (8.6) *below.*

Corollary 1.6 (No constant for a hard cutoff)**.** *For no fixed* $A > 1$ *and no* $C > 0$ *does one have*

$$F(N) \sim CL_A(N).$$

Indeed,

$$h_A(\mathrm{e}^A) = Ah_A(A), \tag{1.12}$$

so h_A *is nonconstant.*

Status of the results. The exact direct-sum theorem, the discriminant estimate, the renewal identity, the slow-variation argument, and the construction of the phase function are proved in this note. The analytic inputs are the classical prime number theorem with a de la Vallée Poussin error term and the previously established two-sided order estimate (1.3); the required upper-bound argument is reproduced with a correction in Section 3. The lower bound is quoted from Bettin–Grenié–Molteni–Sanna.

What is *not* proved is a closed formula or effective numerical table for h_A. The functional equation (1.11) does not determine the function from a scalar normalization: arbitrary data on one fundamental interval propagate to all other scales. We also do not prove continuity, differentiability, or monotonicity of h_A. These points are isolated as open problems in Section 10.

2. Heuristic overview

The original prime-block upper bound uses the elementary fact that an integer $n \le N$ cannot contain two distinct prime factors larger than $N/\log N$. Hence the denominators divisible by a large prime p form a disjoint block

$$B_p = \{p, 2p, \dots, \lfloor N/p \rfloor\, p\},$$

and the subset sums from this block are scaled copies of the smaller problem $E([\lfloor N/p \rfloor])$. Counting the blocks gives a recursive upper bound.

The key point here is that disjointness of the denominator sets is weaker than independence of their *subset-sum values*. Two different choices in one block could conceivably be compensated by choices in other blocks. The right analogue of ordering denominators from largest to second largest is therefore

not ordinary size, but the largest prime-adic level. Fix a large prime p. After clearing every denominator not divisible by p and reducing modulo p, all other blocks disappear. The p-block is forced to close internally.

Internal closure is exactly the injectivity of

$$D_m E([m]) \longrightarrow \mathbb{F}_p.$$

A prime for which this map is injective is separating. For such a prime, equality of two global harmonic subset sums forces equality of the two p-block sums. Peeling off all separating primes gives an exact direct product, not merely an upper envelope.

Why should most primes be separating? Failure means that p divides a nonzero difference between two elements of the finite integer set $D_m E([m])$. Thus every bad prime divides a discriminant-type product. The logarithm of this product is at most $O(mS(m)^2)$. At the polylogarithmic values of m relevant to the large-prime decomposition, the known upper bound implies $S(m) = N^{o(1)}$. Consequently only $N^{o(1)}$ large primes can be bad.

After the bad blocks and the residual set are removed, both contribute $o(N/\log N)$ to $\log S(N)$. The remaining prime blocks give a two-sided asymptotic sum. Grouping primes by $m = \lfloor N/p \rfloor$ produces the kernel

$$\frac{1}{m(m+1)},$$

and normalization converts this to

$$\frac{1}{(m+1)\log m}.$$

The corresponding continuous kernel is $dt/(t \log t)$. Under the change of variables $u = \log_2 t$, it becomes Lebesgue measure. This is the renewal structure behind the iterated logarithms.

The renewal equation yields one-step regular variation,

$$F(N) \sim \log_3 N F(\log N),$$

but a hard terminal cutoff cannot have a constant coefficient. Whenever a new iterated logarithm crosses the cutoff A, the product L_A gains a factor close to A, while $F(N)$ changes negligibly between adjacent integers. The missing coefficient is therefore a phase depending on the last iterate $u_A(N) \in (A, \mathrm{e}^A]$. Its cocycle law

$$h(\mathrm{e}^u) = uh(u)$$

exactly compensates the jump of the hard product.

3. Known bounds and a corrected upper-bound input

We record the analytic inputs and explain a minor point in the upper-bound argument.

Lemma 3.1 (Prime number theorem in the required range)**.** *Let*

$$M_N = \left\lfloor \log N \, (\log_2 N)^{1/2} \right\rfloor .$$

Uniformly for $1 \le m \le M_N$*,*

$$\pi\left(\frac{N}{m}\right) - \pi\left(\frac{N}{m+1}\right) = \frac{N}{m(m+1)\log N}(1+\eta_N(m)), \tag{3.1}$$

where

$$\sup_{m \le M_N} |\eta_N(m)| \ll \frac{\log M_N}{\log N} + M_N \log N \, \mathrm{e}^{-c\sqrt{\log N}} = o(1). \tag{3.2}$$

Proof. Use

$$\pi(x) = \mathrm{li}(x) + O(x\mathrm{e}^{-c_0\sqrt{\log x}}).$$

On the interval $[N/(m+1), N/m]$, one has

$$\frac{1}{\log t} = \frac{1}{\log N}\left(1 + O\left(\frac{\log M_N}{\log N}\right)\right)$$

uniformly for $m \le M_N$. The interval length is $N/(m(m+1))$. Dividing the two endpoint error terms by the main term gives

$$O((m+1)\log N \, \mathrm{e}^{-c\sqrt{\log N}}),$$

which is bounded by the second term in (3.2). □

The prime-block argument of Luo–Yang–Zhu gives, for

$$M(x) := \max_{3 \le n \le x} F(n),$$

a recursion of the form

$$F(N) \le C + (1+\delta(N)) \log_3 N \, M(\log N), \tag{3.3}$$

where

$$\delta(N) \ll \frac{\log_2 N}{\log N} + (\log N)^2 \mathrm{e}^{-c\sqrt{\log N}}. \tag{3.4}$$

Their displayed argument absorbs the factor $1+\delta(N)$ into an additive constant using only $\delta(N)\log_3 N \to 0$. One also needs a bound for $M(\log N)$. The missing estimate is immediate:

$$\begin{aligned} s(n) &\le n \log 2, \\ F(n) &\le (\log 2)\log n, \\ M(\log N) &\le (\log 2)\log_2 N. \end{aligned} \tag{3.5}$$

Hence

$$\delta(N)\log_3 N \, M(\log N) = o(1),$$

and the intended recursion follows:

$$F(N) \le C_1 + \log_3 N \, M(\log N). \tag{3.6}$$

Its standard iteration gives

$$F(N) \ll_A L_A(N). \tag{3.7}$$

For the lower bound we use the following consequence of Bettin–Grenié–Molteni–Sanna: whenever $k \geq 4$ and $\log_k N \geq 3/2$,

$$F(N) \geq 2 \log 2 \left(1 - \frac{3}{2 \log_k N}\right) \prod_{j=3}^{k} \log_j N. \tag{3.8}$$

Together with the elementary comparability of L_A for different fixed thresholds, (3.7) and (3.8) imply (1.3).

We will also use the consequence

$$L_A(x) = (\log_3 x)^{1+o(1)} \tag{3.9}$$

for each fixed $A > 1$.

4. Exact separation of prime blocks

We prove Theorem 1.2.

Proof of Theorem 1.2. Since $q^2 > N$, no integer $n \leq N$ is divisible by two distinct primes larger than q. Thus the blocks B_p are pairwise disjoint.

Suppose that two choices of block sums give the same total:

$$z + \sum_{p \in \mathcal{G}_N} \frac{u_p}{p} = z' + \sum_{p \in \mathcal{G}_N} \frac{v_p}{p}, \tag{4.1}$$

where $z, z' \in E(Z_N)$ and $u_p, v_p \in E([m_p])$. Fix $p \in \mathcal{G}_N$. Write

$$u_p = \frac{a_p}{D_{m_p}}, \qquad v_p = \frac{b_p}{D_{m_p}}, \qquad a_p, b_p \in \mathcal{A}_{m_p}.$$

All terms in (4.1) other than the p-block have denominators prime to p. Indeed, no element of Z_N is divisible by p, and a denominator in a different block has the form $p'j$ with $p' \neq p$ and $j \leq M < p$. Also $p > m_p$, so $p \nmid D_{m_p}$.

After combining the other terms, (4.1) takes the form

$$\frac{a_p - b_p}{p D_{m_p}} + \frac{A}{B} = 0, \qquad p \nmid B.$$

Clearing denominators and reducing modulo p gives

$$(a_p - b_p) B \equiv 0 \pmod{p},$$

so $a_p \equiv b_p \pmod{p}$. Since p is separating for m_p, we obtain $a_p = b_p$, and hence $u_p = v_p$.

This holds for every $p \in \mathcal{G}_N$. Returning to (4.1) then gives $z = z'$. Thus the addition map is injective; surjectivity follows from the partition of $[N]$ into Z_N and the blocks B_p. Equation (1.5) follows. □

5. Bad primes and the residual set

For $m \geq 1$, define the nonzero integer

$$\Delta_m := \prod_{\substack{a,b \in \mathcal{A}_m \\ a<b}} (a-b). \tag{5.1}$$

Lemma 5.1 (Discriminant bound)**.** *If $p > m$ is not separating for m, then $p \mid \Delta_m$. Moreover,*

$$\log |\Delta_m| \ll m S(m)^2. \tag{5.2}$$

Proof. Nonseparation gives distinct $a, b \in \mathcal{A}_m$ with $a \equiv b \pmod{p}$, hence $p \mid a-b$ and $p \mid \Delta_m$.

Every element of $\mathcal{A}_m$ lies in $[0, D_m H_m]$, where $H_m = \sum_{n \leq m} 1/n$. Therefore

$$\log |\Delta_m| \leq \binom{S(m)}{2} \log(D_m H_m).$$

The Chebyshev estimate $\log D_m \ll m$ and $\log H_m \ll \log_2 m$ give (5.2). □

Set from now on

$$M = M_N := \left\lfloor \log N (\log_2 N)^{1/2} \right\rfloor, \qquad q := \frac{N}{M+1}. \tag{5.3}$$

For large N, $M < q$ and $q^2 > N$.

Lemma 5.2 (Scarcity of bad large primes)**.** *Let $\mathcal{B}_N$ be the primes $p > q$ that are not separating for $m_p = \lfloor N/p \rfloor$. Then*

$$|\mathcal{B}_N| = N^{o(1)}. \tag{5.4}$$

Proof. For a fixed $m \leq M$, every bad prime in

$$\left(\frac{N}{m+1}, \frac{N}{m} \right]$$

divides Δ_m and is at least q. Hence the number of such primes is at most

$$\frac{\log |\Delta_m|}{\log q} \ll \frac{m S(m)^2}{\log q}.$$

Since $S(m)$ is nondecreasing,

$$|\mathcal{B}_N| \ll \frac{1}{\log q} \sum_{m \leq M} m S(m)^2 \ll \frac{M^2 S(M)^2}{\log q}. \tag{5.5}$$

By (3.7), (3.9), and the definition of M,

$$\begin{aligned} s(M) &\ll \frac{M}{\log M} L_A(M) \\ &\ll \log N \, (\log_2 N)^{-1/2} (\log_4 N)^{1+o(1)} = o(\log N). \end{aligned}$$

Thus $S(M) = N^{o(1)}$, and (5.5) proves the claim. □

Let Z_N be the residual set in Theorem 1.2, with $\mathcal{G}_N$ the separating primes above q.

Lemma 5.3 (Residual contribution)**.** *One has*

$$\log |E(Z_N)| = o\left(\frac{N}{\log N}\right). \tag{5.6}$$

Moreover,

$$\sum_{p\in\mathcal{B}_N} s(m_p) = o\left(\frac{N}{\log N}\right). \tag{5.7}$$

Proof. Let $d_N = \operatorname{lcm}(Z_N)$. Every prime factor of an element of Z_N that exceeds q belongs to $\mathcal{B}_N$. Since $q^2 > N$, such a prime occurs only to the first power. Therefore

$$\log d_N \le \vartheta(q) + \sum_{k\ge 2} \vartheta(N^{1/k}) + |\mathcal{B}_N| \log N. \tag{5.8}$$

The first two terms are

$$O(q + \sqrt{N} + N^{1/3}\log N) = o\left(\frac{N}{\log N}\right),$$

and the last term has the same property by (5.4). Thus

$$\log d_N = o(N/\log N).$$

Every element of $E(Z_N)$ has the form a/d_N with $0 \le a \le d_N H_N$. Hence

$$|E(Z_N)| \le d_N H_N + 1,$$

which proves (5.6).

Finally, $s(m_p) \le m_p \log 2 \le M \log 2$, and therefore

$$\sum_{p\in\mathcal{B}_N} s(m_p) \le |\mathcal{B}_N|\, M \log 2 = N^{o(1)} = o(N/\log N).$$

□

Combining Theorems 1.2 and 5.3 gives the two-sided prime recursion

$$s(N) = \sum_{p>q} s\left(\left\lfloor \frac{N}{p} \right\rfloor\right) + o\left(\frac{N}{\log N}\right), \tag{5.9}$$

where the sum is over primes.

6. The asymptotic renewal identity

Proof of Theorem 1.3. For $1 \le m \le M$, write

$$a_m(N) := \#\left\{p : \frac{N}{m+1} < p \le \frac{N}{m}\right\}.$$

By Theorem 3.1,

$$a_m(N) = \frac{N}{m(m+1)\log N}(1 + \eta_N(m))$$

uniformly for $m \le M$. Equation (5.9) gives

$$F(N) = \sum_{m \le M} \frac{s(m)}{m(m+1)} + o(1). \tag{6.1}$$

Indeed, the total prime-number-theorem error is bounded by

$$\sup_{m \le M} |\eta_N(m)| \sum_{m \le M} \frac{s(m)}{m(m+1)} \le (\log 2) \sup_{m \le M} |\eta_N(m)| \sum_{m \le M} \frac{1}{m+1} = o(1).$$

It remains to remove $\log N < m \le M$. By (3.7) and monotonicity of L_A,

$$\begin{aligned} \sum_{\log N < m \le M} \frac{s(m)}{m(m+1)} &\ll \frac{L_A(M)}{\log_2 N} \sum_{\log N < m \le M} \frac{1}{m} \\ &\ll \frac{L_A(M) \log_3 N}{\log_2 N} = o(1), \end{aligned}$$

where the last step follows from (3.9) applied at M. Consequently,

$$F(N) = \sum_{m \le \log N} \frac{s(m)}{m(m+1)} + o(1).$$

Since $s(1) = \log 2$, $s(2) = 2 \log 2$, and

$$\frac{s(m)}{m(m+1)} = \frac{F(m)}{(m+1)\log m} \qquad (m \ge 3),$$

we obtain (1.6). □

7. Continuous renewal and slow variation

Define the step extension

$$\widetilde{F}(x) := F(\lfloor x \rfloor), \qquad x \ge 3,$$

and put

$$a := \log_2 3, \qquad G(z) := \widetilde{F}(\mathrm{e}^{\mathrm{e}^z}). \tag{7.1}$$

For $m \ge 3$, let

$$d_m := \frac{1}{(m+1)\log m} - \int_m^{m+1} \frac{dt}{t \log t}. \tag{7.2}$$

Since

$$d_m = O\left(\frac{1}{m^2 \log m}\right), \qquad F(m) \le (\log 2) \log m,$$

the series $\sum_{m \ge 3} F(m) d_m$ converges absolutely.

Lemma 7.1 (Continuous renewal equation)**.** *There is a real constant*

$$\gamma_F := \frac{5 \log 2}{6} + \sum_{m=3}^{\infty} F(m) d_m \tag{7.3}$$

and a function $r(z) \to 0$ such that

$$G(z) = \gamma_F + \int_a^{\log z} G(u)\, du + r(z). \tag{7.4}$$

Proof. Replace the sum in (1.6) by the integral over the unit intervals $[m, m+1)$. The accumulated difference tends to the absolutely convergent series in (7.3). The change of variables $u = \log_2 t$ gives $dt/(t \log t) = du$. The floor and endpoint discrepancies tend to zero by the trivial bound $F(x) \ll \log x$. □

For fixed $A > 1$, define

$$\Lambda_A(z) := \prod_{j=1}^{\nu_A(z)} \log_j z, \qquad \nu_A(z) := \max(\{j \geq 1 : \log_j z > A\} \cup \{0\}). \tag{7.5}$$

Then (1.3) is equivalent to

$$G(z) \asymp_A \Lambda_A(z). \tag{7.6}$$

Lemma 7.2 (Slow variation)**.** *For every fixed $\lambda > 0$,*

$$\frac{G(\lambda z)}{G(z)} \longrightarrow 1. \tag{7.7}$$

Proof. It suffices to consider $\lambda > 1$. Subtracting (7.4) at z and λz gives

$$G(\lambda z) - G(z) = \int_{\log z}^{\log z + \log \lambda} G(u)\, du + o(1).$$

For u in this bounded interval, (7.6) and the identity

$$\Lambda_A(z) = \log z\, \Lambda_A(\log z)$$

for sufficiently large z imply

$$G(u) \ll_{A,\lambda} \frac{G(z)}{\log z}.$$

Since $G(z) \to \infty$, division by $G(z)$ proves (7.7). The case $0 < \lambda < 1$ follows by inversion. □

We use the elementary Karamata consequence of slow variation.

Lemma 7.3 (Karamata integral relation)**.** *One has*

$$\int_a^x G(t)\, dt \sim xG(x). \tag{7.8}$$

Proof. This is the standard integral theorem for a positive slowly varying function. For completeness, fix $0 < \delta < 1$. The uniform convergence theorem for slowly varying functions gives

$$\int_{\delta x}^x G(t)\, dt = xG(x) \int_\delta^1 \frac{G(xs)}{G(x)}\, ds \sim (1-\delta) xG(x).$$

Potter's bound controls the remaining interval $[a, \delta x]$ by $O(\delta^{1/2} xG(x))$. Letting $\delta \downarrow 0$ proves the claim. □

Proof of Theorem 1.4. By Theorems 7.1 and 7.3,

$$G(z) \sim \log z \, G(\log z).$$

Returning to F gives (1.7). Iterating a fixed number of times proves (1.8). □

8. Second-order renewal and the phase function

The existence of a tower limit requires a summable one-step error, not merely a ratio tending to one. The next lemma supplies it.

Lemma 8.1 (Second-order integral asymptotic)**.** *As $x \to \infty$,*

$$xG(x) - \int_a^x G(t)\,dt \sim xG(\log x). \tag{8.1}$$

Proof. Choose b so large that (7.4) is valid on $[b, \infty)$ and $\log b > a$. Write

$$D_b(x) := (x-b)G(x) - \int_b^x G(t)\,dt = \int_b^x \big(G(x) - G(t)\big)\,dt.$$

Subtracting (7.4) at x and t gives

$$G(x) - G(t) = \int_{\log t}^{\log x} G(u)\,du + r(x) - r(t).$$

The contribution of the r-terms is $o(x)$. Since $G(\log x) \to \infty$, this is negligible compared with $xG(\log x)$. Fubini's theorem yields

$$\begin{aligned} D_b(x) &= \int_{\log b}^{\log x} (\mathrm{e}^u - b)G(u)\,du + o(xG(\log x)) \\ &= \int_{\log b}^{\log x} \mathrm{e}^u G(u)\,du + o(xG(\log x)). \end{aligned}$$

The discarded term $b\int_{\log b}^{\log x} G(u)\,du$ is $O(G(x)) = o(xG(\log x))$ by Theorem 1.4.

It remains to evaluate the exponentially weighted integral. Put $Y = \log x$ and write

$$\frac{1}{\mathrm{e}^Y G(Y)} \int_{\log b}^{Y} \mathrm{e}^u G(u)\,du = \int_0^{Y-\log b} \mathrm{e}^{-v} \frac{G(Y-v)}{G(Y)}\,dv.$$

For each fixed v, slow variation gives $G(Y-v)/G(Y) \to 1$. The comparability (7.6) and monotonicity of Λ_A give an integrable domination by $C\mathrm{e}^{-v}$. Dominated convergence therefore gives

$$\int_{\log b}^{Y} \mathrm{e}^u G(u)\,du \sim \mathrm{e}^Y G(Y) = xG(\log x).$$

The finite lower interval $[a, b]$ changes (8.1) only by $O(G(x))$, which is negligible. □

Define

$$R(x) := \frac{G(\mathrm{e}^x)}{xG(x)}. \tag{8.2}$$

Corollary 8.2 (Summable one-step error)**.** *One has*

$$R(x) = 1 - \frac{1}{\log x} + o\left(\frac{1}{\log x}\right). \tag{8.3}$$

Proof. By (7.4),

$$G(\mathrm{e}^x) = \gamma_F + \int_a^x G(t)\,dt + r(\mathrm{e}^x).$$

Apply Theorem 8.1 and divide by $xG(x)$. Finally, Theorem 1.4 gives

$$\frac{G(\log x)}{G(x)} \sim \frac{1}{\log x}.$$

□

Define the exponential tower

$$E_0(u) = u, \qquad E_{j+1}(u) = \mathrm{e}^{E_j(u)}, \qquad P_r(u) := \prod_{j=0}^{r-1} E_j(u), \quad P_0(u) = 1. \tag{8.4}$$

Proposition 8.3 (Tower limit)**.** *For every $u > 1$, the limit*

$$h(u) := \lim_{r\to\infty} \frac{G(E_r(u))}{P_r(u)} \tag{8.5}$$

exists and is positive. The convergence is uniform for u in every compact subinterval of $(1, \infty)$. Moreover,

$$h(u) = G(u) \prod_{j=0}^{\infty} R(E_j(u)). \tag{8.6}$$

Proof. Set

$$H_r(u) := \frac{G(E_r(u))}{P_r(u)}.$$

Then

$$\frac{H_{r+1}(u)}{H_r(u)} = R(E_r(u)).$$

By Theorem 8.2, for $r \geq 1$,

$$R(E_r(u)) = 1 + O\left(\frac{1}{E_{r-1}(u)}\right),$$

uniformly for u in a fixed compact subset of $(1, \infty)$. The series

$$\sum_{r\geq 1} \frac{1}{E_{r-1}(u)}$$

converges uniformly on such compact sets. Therefore the infinite product in (8.6) converges uniformly to a positive limit, proving the result. □

Lemma 8.4 (Cocycle identity)**.** *The function h satisfies*

$$h(\mathrm{e}^u) = uh(u)$$

for every $u > 1$.

Proof. Using $E_r(\mathrm{e}^u) = E_{r+1}(u)$ and

$$P_r(\mathrm{e}^u) = \prod_{j=1}^{r} E_j(u) = \frac{P_{r+1}(u)}{u},$$

we obtain

$$h(\mathrm{e}^u) = \lim_{r\to\infty} \frac{G(E_{r+1}(u))}{P_{r+1}(u)/u} = uh(u).$$

□

Proof of Theorem 1.5. Let

$$z = \log_2 N, \qquad k = \kappa_A(N) - 2, \qquad u = u_A(N) = \log_k z.$$

Then $u \in (A, \mathrm{e}^A]$, $z = E_k(u)$, and

$$L_A(N) = \prod_{j=3}^{\kappa_A(N)} \log_j N = P_k(u).$$

Thus

$$\frac{F(N)}{L_A(N)} = \frac{G(E_k(u))}{P_k(u)} + o(1).$$

Since $k \to \infty$ and the convergence in Theorem 8.3 is uniform on $[A, \mathrm{e}^A]$, this tends uniformly to $h(u)$. Set

$$h_A = h|_{[A,\mathrm{e}^A]}.$$

This proves (1.10); (1.11) follows from Theorem 8.4. □

Proof of Theorem 1.6. Equation (1.12) is (1.11) with $u = A$, so h_A is non-constant. To rule out a constant asymptotic without imposing any continuity on h_A, use the jumps of the hard cutoff directly.

For $r \geq 3$, let $x_r = E_r(A)$ and $n_r = \lceil x_r \rceil$. Then

$$\log_r(n_r - 1) \leq A < \log_r n_r.$$

All earlier iterated logarithms change by a relative $o(1)$ between $n_r - 1$ and n_r, whereas the new terminal factor satisfies $\log_r n_r \to A$. Hence

$$\frac{L_A(n_r)}{L_A(n_r - 1)} \longrightarrow A.$$

On the other hand,

$$S(N) \leq S(N+1) \leq 2S(N),$$

so $0 \leq s(N+1) - s(N) \leq \log 2$. Since $s(N) \to \infty$, it follows that

$$\frac{F(n_r)}{F(n_r - 1)} \longrightarrow 1.$$

Therefore the quotient $F(N)/L_A(N)$ changes by a factor tending to $1/A$ across these adjacent threshold integers. It cannot converge to a positive constant. □

9. What the phase formula computes, and what it does not

The phase is rigorously determined by the arithmetic function F through either (8.5) or (8.6). The product tail converges extremely rapidly on a tower scale. More precisely, Theorem 8.2 implies that, for fixed $A > 1$ and sufficiently large J,

$$h_A(u) = \frac{G(E_J(u))}{P_J(u)} \left(1 + O_A\left(\frac{1}{E_{J-1}(u)}\right)\right) \tag{9.1}$$

uniformly for $u \in [A, \mathrm{e}^A]$.

The scalar γ_F in (7.3) is also computable from finite values of $S(m)$ with an absolutely convergent tail. Nevertheless, γ_F alone does not determine h_A. The renewal relation is a variable-delay equation: the values on one exponential layer depend on the entire profile on the previous layer. Equivalently, the cocycle equation (1.11) permits arbitrary positive data on a fundamental interval, subject only to endpoint compatibility. The actual arithmetic selects one such profile.

The lower bound (3.8) gives an explicit pointwise constraint. Taking $N = E_{r+2}(u)$ and $k = r + 2$, then letting $r \to \infty$, yields

$$h(u) \geq 2 \log 2 \left(1 - \frac{3}{2u}\right) \qquad (u \geq 3/2). \tag{9.2}$$

For the standard cutoff $A = 2$, the cocycle identity improves the left endpoint to

$$h_2(2) = \frac{1}{2} h_2(\mathrm{e}^2) \geq \log 2 \left(1 - \frac{3}{2\mathrm{e}^2}\right) = 0.552436\ldots. \tag{9.3}$$

The general upper estimate in (1.3) gives $h_A(u) \ll_A 1$.

A rigorous high-precision numerical plot would require two additional effective ingredients:

(i) an explicit error bound in (1.6), including constants in the bad-prime estimate and the prime-number-theorem remainder;
(ii) sufficiently many exact or certified values of $S(m)$ to initialize the method of steps.

Neither ingredient is supplied by the present argument at a practical numerical scale. Accordingly, this note does not claim a numerical evaluation or a closed form for h_A.

10. Open problems

Question 10.1 (Effective phase computation)**.** *Obtain an explicit uniform error term in (1.6) strong enough to compute h_2 with certified numerical error. Even a rigorous plot on $[2, \mathrm{e}^2]$ would substantially clarify the terminal oscillation.*

Question 10.2 (Regularity of the phase)**.** *Is h continuous, Hölder continuous, or differentiable? Does it have any monotonicity or convexity on a fundamental interval? The present proof gives positivity, boundedness, uniform tower convergence, and the cocycle identity, but no finer regularity.*

Question 10.3 (Closed form or canonical equation)**.** *Can h be characterized as the unique solution of a closed renewal or delay equation with a natural normalization? The scalar renewal constant γ_F is insufficient by itself; one needs a condition that captures the arithmetic initial profile.*

Question 10.4 (Sharper bad-prime estimates)**.** *The discriminant argument gives only $N^{o(1)}$ bad large primes. Is the number of bad primes bounded, polylogarithmic, or even eventually zero in the relevant intervals? Structural information on the difference set $\mathcal{A}_m - \mathcal{A}_m$ could lead to a substantially more effective renewal error.*

Question 10.5 (Exact local states beyond separating primes)**.** *For a bad prime, reduction modulo p merges several elements of $\mathcal{A}_m$. Can one retain a finite collision state and obtain an exact matrix-valued prime-block recursion rather than discarding the block? Such a refinement may encode h_A more explicitly.*

Question 10.6 (Dissociated denominators)**.** *Let $R(N)$ be the largest cardinality of $A \subset [N]$ whose reciprocal subset sums are all distinct. The known estimates give $R(N) \asymp \log S(N)$. Does $R(N)$ possess a phase-resolved asymptotic with the same h_A, or with a different terminal profile?*

Question 10.7 (Collision entropy)**.** *For*

$$r_N(x) = \#\left\{A \subset [N] : \sum_{n\in A} \frac{1}{n} = x\right\},$$

study the collision quantities $\sum_x r_N(x)^q$. Separating prime blocks give exact multiplicativity at the level of the full representation function. Do the corresponding Rényi entropies satisfy analogous renewal equations and phase laws?

Question 10.8 (Other coefficient sets and weights)**.** *Extend the method to coefficients in $\{0, 1, \dots, r\}$, signed coefficients, or weighted reciprocal sequences. Which local modular separation conditions produce the same $dt/(t \log t)$ renewal kernel, and when does a different iterated-logarithmic hierarchy appear?*

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manuscript was developed with substantial AI assistance; the named author assumes responsibility for checking the arguments and for any remaining errors.

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