REFLECTION COMPLETIONS AND WEIGHTED STABILITY
FOR STRONGLY 2-PRIMITIVE SETS

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Abstract. Write

$$Y = n^{1/3}, \qquad M = \frac{Y}{\log n}, \qquad S = M^2 = \frac{n^{2/3}}{(\log n)^2}.$$

Chojecki proves [1] that the largest strongly 2-primitive subset of $[1, n]$ has size

$$\pi(n) + \left(\frac{27}{2} + o(1)\right) S.$$

This draft develops two further statements.

First, we give a complete discretization of a *reflection completion*

$$z = \frac{2}{\sqrt{x}} - y, \qquad 0 < x < y < \frac{1}{\sqrt{x}},$$

using transformed prime bins and proper edge-colourings. It produces a linear prime-triple packing with the full constant $27/2$, but with a positive-density subfamily having any prescribed fixed multiplicative slack. Consequently, for every fixed integer $m \geq 2$, there are reduced normalized near-extremizers containing

$$\left[\frac{27}{2}\left(1 - \sqrt{1 - \frac{1}{m}}\right)^{4/3} - o(1)\right] S$$

residual elements of the form $mpqr$. In particular, neither squarefreeness nor support three is forced.

Second, assuming the main theorem and the private-prime normalization theorem from [3], we prove a weighted normal form. After removing $o(S)$ residual elements, every normalized near-extremizer is a family

$$\left\{\lambda_E \prod_{p \in E} p : E \in \mathcal{H}_n\right\},$$

where $\mathcal{H}_n$ is a linear 3-uniform hypergraph of size $(27/2 - o(1))S$, all its vertex primes are $n^{1/3+o(1)}$, and $\lambda_E = n^{o(1)}$. All prime divisors of all multipliers lie in a common hub of size $n^{o(1)}$. Thus the correct replacement for "predominantly squarefree triples" is "predominantly small-hub weighted linear triples."

The arguments below are written as a proof draft and should be independently checked before being treated as a final theorem.

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Contents

1. Setup and hypotheses

A set $A \subseteq [1, n]$ is *strongly* 2*-primitive* if

$$a \nmid bc \qquad (a, b, c \in A,\ a \notin \{b, c\}),$$

where $b = c$ is allowed. We use the following two results from [1, 3]. This condition is closely related to the multiplicative sequence problem initiated by Erdős [2].

Main asymptotic hypothesis. If $F(n)$ denotes the largest cardinality of such a set, then

$$F(n) = \pi(n) + \left(\frac{27}{2} + o(1)\right) S. \tag{1}$$

Normalized stability hypothesis. If

$$|A| = \pi(n) + \left(\frac{27}{2} - o(1)\right) S, \tag{2}$$

then private-prime contractions produce a strongly 2-primitive set of the same size having the form

$$\widetilde{A} = (\mathbb{P} \cap [1,n] \setminus V) \cup C, \qquad \operatorname{supp}(c) \subseteq V \quad (c \in C), \tag{3}$$

with

$$|V| = o(S), \qquad |C| = \left(\frac{27}{2} + o(1)\right) S. \tag{4}$$

The same note proves that all but $o(S)$ elements of C have at least three distinct prime divisors.

The purpose of the present draft is to replace the unproved squarefree-triple conclusion by a weighted triple theorem, and to construct examples showing that the weights cannot be removed.

2. Continuous reflection geometry

Put

$$\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2_{>0} : 0 < x < y < \frac{1}{\sqrt{x}}\right\}. \tag{5}$$

For $(x,y) \in \mathcal{D}$, define

$$z_0 = z_0(x,y) = \frac{2}{\sqrt{x}} - y. \tag{6}$$

Then $x < y < z_0$ and

$$xyz_0 = xy\left(\frac{2}{\sqrt{x}} - y\right) = t(2-t), \qquad t = y\sqrt{x} \in (0,1). \tag{7}$$

In particular $xyz_0 \le 1$.

2.1. **The three roles of a shared pair.** Define

$$\psi(u) = u^{-1/2}, \qquad \tau(u) = 2u^{-1/2} - u.$$

For $0 < u < v$, away from the two boundary curves, classify the pair (u,v) as

$$\begin{aligned} \text{lower} &\iff v < \psi(u), \\ \text{outer} &\iff \psi(u) < v < \tau(u), \\ \text{upper} &\iff v > \tau(u). \end{aligned}$$

If $u \ge 1$, then $v > u \ge \tau(u)$, so the pair is automatically upper. If $u < 1$, then $\psi(u) < \tau(u)$, so these three open regions are disjoint.

Lemma 2.1 (Role lemma)**.** *Let $(x,y) \in \mathcal{D}$ and z_0 be given by (6). Then*

$$(x,y) \textit{ is lower}, \qquad (x,z_0) \textit{ is outer}, \qquad (y,z_0) \textit{ is upper}.$$

The same conclusion holds if z_0 is replaced by any $z < z_0$ sufficiently close to z_0, uniformly on a compact subset of $\mathcal{D}$.

Proof. The first assertion is the defining inequality $y < x^{-1/2}$. Next,

$$z_0 - x^{-1/2} = x^{-1/2} - y > 0, \qquad \tau(x) - z_0 = y - x > 0,$$

so (x, z_0) is outer. Finally,

$$z_0 - \tau(y) = \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{y}} > 0,$$

so (y, z_0) is upper. On a compact subset of $\mathcal{D}$, the three positive gaps above have a positive minimum, which gives the uniform perturbation statement. □

2.2. **Area and slack.** The area of the feasible domain is

$$\operatorname{area}(\mathcal{D}) = \int_0^1 \left(x^{-1/2} - x\right) dx = \frac{3}{2}. \tag{8}$$

For a fixed integer $m \geq 1$, let

$$t_m = 1 - \sqrt{1 - \frac{1}{m}}. \tag{9}$$

Since $t \mapsto t(2-t)$ is increasing on $(0,1)$, the reflected product in (7) is at most $1/m$ precisely when $t \leq t_m$. Set

$$\mathcal{D}_m = \{(x, y) \in \mathcal{D} : y\sqrt{x} < t_m\}.$$

A direct integration gives

$$\operatorname{area}(\mathcal{D}_m) = \int_0^{t_m^{2/3}} \left(t_m x^{-1/2} - x\right) dx = \frac{3}{2} t_m^{4/3}. \tag{10}$$

3. A rigorous discretized reflection packing

3.1. **The two coordinate systems.** Fix $h > 0$. For an integer j, put

$$J_j = (jh, (j+1)h].$$

The ordinary prime bin is

$$Q_j(n) = \{q \in \mathbb{P} : q/Y \in J_j\}. \tag{11}$$

For the first prime we bin the transformed coordinate

$$s(x) = \frac{2}{\sqrt{x}}.$$

For an integer $i \geq 1$, let

$$\begin{aligned} X_i &= \{x > 0 : s(x) \in (ih, (i+1)h]\} \\ &= \left[\frac{4}{(i+1)^2 h^2}, \frac{4}{i^2 h^2}\right), \end{aligned} \tag{12}$$

and define

$$P_i(n) = \{p \in \mathbb{P} : p/Y \in X_i\}. \tag{13}$$

For a cell (i, j) put

$$k = i - j - 2. \tag{14}$$

If $x \in X_i$, $y \in J_j$, and $z \in J_k$, then

$$y + z \le (j + 1 + k + 1)h = ih < s(x). \tag{15}$$

Moreover, if $z_0 = s(x) - y$, then

$$0 < z_0 - z < 3h. \tag{16}$$

Thus z lies just below the exact reflection point.

3.2. **Safe cells.** Let $K \Subset \mathcal{D}$ be compact with boundary of planar measure zero. For small h, let $\mathcal{C}_h(K)$ be the finite collection of pairs (i, j) such that

$$\overline{X_i} \times \overline{J_j} \subset K.$$

For each such pair define k by (14).

Lemma 3.1 (Uniform safety)**.** *For every compact $K \Subset \mathcal{D}$, there exists $h_0(K) > 0$ such that, whenever $0 < h < h_0(K)$, $(i, j) \in \mathcal{C}_h(K)$, and*

$$x \in X_i, \qquad y \in J_j, \qquad z \in J_{i-j-2},$$

one has

$$x < y < z, \qquad xyz < 1, \tag{17}$$

and the three pairs $(x, y), (x, z), (y, z)$ have the lower, outer, and upper roles, respectively.

Proof. On K, the continuous functions

$$y - x, \qquad x^{-1/2} - y, \qquad 2x^{-1/2} - 2y^{-1/2}$$

have a positive common lower bound, say δ. By (16), z differs from z_0 by less than $3h$. Taking $h < \delta/6$ and applying Theorem 2.1 proves the ordering and role statements.

By (15) and the arithmetic–geometric mean inequality,

$$4yz \le (y + z)^2 < s(x)^2 = \frac{4}{x},$$

which gives $xyz < 1$. □

3.3. **Prime counts and colour capacity.** Write

$$\Delta_i = |X_i| = \frac{4}{i^2h^2} - \frac{4}{(i + 1)^2h^2}. \tag{18}$$

For a fixed finite collection of cells, the prime number theorem gives uniformly

$$|P_i(n)| = (3 + o(1))M\Delta_i, \qquad |Q_j(n)| = (3 + o(1))Mh. \tag{19}$$

Because $K \Subset \mathcal{D}$, its x-projection lies in $(0, 1 - \eta]$ for some $\eta > 0$. Since $x = 4/s^2$ and $|dx/ds| = x^{3/2}$,

$$\Delta_i \le (1 - \eta)^{3/2}h \tag{20}$$

for every cell meeting K. Hence, for all sufficiently large n,

$$|P_i(n)| < \min\{|Q_j(n)|, |Q_k(n)|\} \tag{21}$$

up to an $o(M)$ discrepancy between the two ordinary bins.

For a cell (i,j), choose a subset

$$Q_{ij} \subseteq Q_j(n), \qquad |Q_{ij}| = \min\{|Q_j(n)|, |Q_k(n)|\}.$$

For large n, the complete bipartite graph

$$K_{P_i(n),Q_{ij}}$$

can be properly edge-coloured with $|Q_{ij}|$ colours, and these colours can be injected into $Q_k(n)$. We use a cyclic colouring, so every used colour class has exactly $|P_i(n)|$ edges.

Each edge $\{p,q\}$ coloured by $r \in Q_k(n)$ creates the prime triple $\{p,q,r\}$.

3.4. **Global linearity.**

Proposition 3.2 (No leakage between reflection cells)**.** *For fixed $K \Subset \mathcal{D}$ and sufficiently small h, the union of the triples constructed from all cells in $\mathcal{C}_h(K)$ is a linear 3-uniform hypergraph.*

Proof. Within one cell, the lower pair $\{p,q\}$ is an edge of a simple bipartite graph. A repeated pair $\{p,r\}$ or $\{q,r\}$ would repeat a colour at one endpoint, contrary to properness.

Now suppose two triples from possibly different cells share a pair of primes. By Theorem 3.1, the numerical values of the shared pair determine whether it is lower, outer, or upper; this role cannot change between the two triples.

If the pair is lower, its transformed first-prime bin i and ordinary second-prime bin j are fixed, hence $k = i - j - 2$ is fixed. If the pair is outer, i and k are fixed, hence $j = i - k - 2$ is fixed. If the pair is upper, j and k are fixed, hence $i = j + k + 2$ is fixed. Thus both triples came from the same cell, where the first paragraph applies. □

3.5. **Counting the packing.**

The rectangles $X_i \times J_j$ form a mesh whose diameter tends to zero on compact subsets. Hence

$$\sum_{(i,j)\in\mathcal{C}_h(K)} \Delta_i h \longrightarrow \operatorname{area}(K) \qquad (h \to 0). \tag{22}$$

By (19), the cell (i,j) contributes

$$|P_i(n)|\,|Q_{ij}| = (9 + o(1))M^2 \Delta_i h \tag{23}$$

triples.

Theorem 3.3 (Reflection packing with prescribed slack)**.** *For every fixed integer $m \geq 1$, there exist linear families $\mathcal{H}_n$ of triples of distinct primes and* *subfamilies* $\mathcal{H}_n^{(m)} \subseteq \mathcal{H}_n$ *such that*

$$\prod_{p\in E} p \le n \qquad (E \in \mathcal{H}_n), \tag{24}$$

$$m\prod_{p\in E} p \le n \qquad (E \in \mathcal{H}_n^{(m)}), \tag{25}$$

$$|\mathcal{H}_n| = \left(\frac{27}{2} - o(1)\right) S, \tag{26}$$

$$|\mathcal{H}_n^{(m)}| = \left[\frac{27}{2}\left(1 - \sqrt{1 - \frac{1}{m}}\right)^{4/3} - o(1)\right] S, \tag{27}$$

$$|V(\mathcal{H}_n)| = o(S). \tag{28}$$

The construction can be arranged so that every used vertex has degree tending to infinity.

Proof. Fix $\varepsilon > 0$. Choose compact Jordan-measurable sets

$$K_m \Subset K \Subset \mathcal{D}$$

with

$$\operatorname{area}(K) > \frac{3}{2} - \varepsilon, \qquad \operatorname{area}(K_m) > \frac{3}{2} t_m^{4/3} - \varepsilon, \qquad K_m \subset \mathcal{D}_m.$$

Take h small enough that the inner-cell Riemann sums for K and K_m are within ε of their areas. Construct the triples from $\mathcal{C}_h(K)$, and mark those cells lying inside K_m.

Linearity follows from Theorem 3.2. The product bound follows from Theorem 3.1. For a marked cell, $z < z_0$ and $(x, y) \in \mathcal{D}_m$, so

$$xyz < xyz_0 = t(2 - t) < \frac{1}{m},$$

which proves (25). Equations (19)–(22) give the two edge counts, with the factor 9 multiplying the areas in (8) and (10).

Only finitely many prime bins are used for fixed K, h, so the vertex set has size $O_{K,h}(M) = o(S)$. In the cyclic colourings, every endpoint and every used colour has degree comparable to M, hence tending to infinity.

Finally let $\varepsilon \to 0$ by a standard diagonal choice of K, h as $n \to \infty$. □

Corollary 3.4 (Positive-density multiplier layers)**.** *For every fixed integer* $m \ge 2$*, there are strongly* 2*-primitive sets* $A_n \subseteq [1, n]$ *satisfying*

$$|A_n| = \pi(n) + \left(\frac{27}{2} - o(1)\right) S$$

and containing

$$\left[\frac{27}{2}\left(1 - \sqrt{1 - \frac{1}{m}}\right)^{4/3} - o(1)\right] S$$

residual composites of the form $mpqr$ with p,q,r distinct primes. The examples remain unchanged after all private-prime contractions.

Proof. Take $\mathcal{H}_n$ and $\mathcal{H}_n^{(m)}$ from Theorem 3.3. For large n, the fixed prime divisors of m are disjoint from $V(\mathcal{H}_n)$. Define

$$\begin{aligned} A_n = &\big(\mathbb{P}\cap[1,n]\setminus(V(\mathcal{H}_n)\cup\operatorname{supp}(m))\big) \\ &\cup\left\{\prod_{p\in E} p : E\in\mathcal{H}_n\setminus\mathcal{H}_n^{(m)}\right\} \\ &\cup\left\{m\prod_{p\in E} p : E\in\mathcal{H}_n^{(m)}\right\}. \end{aligned}$$

For a target edge E, two other edges contain at most two of its three vertex primes, while the multipliers use only the fixed hub primes in $\operatorname{supp}(m)$. Thus the target cannot divide their product. Retained singleton primes divide no composite. Hence A_n is strongly 2-primitive.

The size is

$$\pi(n) - |V(\mathcal{H}_n)| - |\operatorname{supp}(m)| + |\mathcal{H}_n|,$$

which has the required asymptotic. Every hypergraph vertex has degree tending to infinity, and every prime dividing m occurs in $|\mathcal{H}_n^{(m)}|\to\infty$ composites. Thus no prime used by the residual core is globally private. $\square$

Remark 3.5. For $m=2$, the constant in Theorem 3.4 is

$$\frac{27}{2}\left(1-\frac{1}{\sqrt{2}}\right)^{4/3} = 2.625908\ldots.$$

Taking $m=4$ gives a positive-density family $4pqr = 2^2pqr$, so repeated prime exponents also survive normalized reduction.

4. Saturation of weighted feasible-pair slots

Define the canonical feasible-pair set

$$\mathcal{D}_n = \{(p,q) : p<q \text{ primes and } pq^2\le n\}. \tag{29}$$

Chojecki proves [1]

$$|\mathcal{D}_n| = \left(\frac{27}{2}+o(1)\right)S. \tag{30}$$

We now show that pair saturation is already unconditional at the level of the private-factor certificate; no squarefree-triple hypothesis is needed.

Let

$$\begin{aligned}
B_0 &= [1, n^{3/5}],\\
B_1 &= \{p \in \mathbb{P} : n^{3/5} < p \le n\},\\
B_2 &= \{pq : p, q \le Y \text{ primes}\},\\
B_3 &= \{qr : Y < q \le n^{2/5},\ r \le n/q^2,\ q, r \text{ primes}\}.
\end{aligned}$$

The multiplicative-basis proof chooses for every $a \le n$ a factorization

$$a = u_a v_a, \qquad u_a \in B_0, \qquad v_a \in B_0 \cup B_1 \cup B_2 \cup B_3. \tag{31}$$

The factor graph of a strongly 2-primitive family is a star forest, and near equality implies that all but $o(S)$ vertices outside B_0 have degree one.

Theorem 4.1 (Weighted feasible-pair saturation)**.** *Let $\widetilde{A}$ have the normalized form* (3) *and satisfy* (2)*. There exist*

$$C_0 \subseteq C, \qquad E_n \subseteq \mathcal{D}_n,$$

and a bijection $C_0 \leftrightarrow E_n$ such that

$$|C \setminus C_0| = o(S), \tag{32}$$

$$|\mathcal{D}_n \setminus E_n| = o(S), \tag{33}$$

and, if $c_{p,q} \in C_0$ corresponds to $(p, q) \in E_n$, then

$$c_{p,q} = pq\, w_{p,q}, \qquad 1 \le w_{p,q} \le \frac{n}{pq}, \qquad \operatorname{supp}(c_{p,q}) \subseteq V. \tag{34}$$

For every fixed $\varepsilon > 0$, all but $o(S)$ pairs in E_n satisfy

$$n^{1/3-\varepsilon} < p < q < n^{1/3+\varepsilon}. \tag{35}$$

Proof. Let

$$W = (B_2 \cup B_3) \setminus B_0.$$

The secondary counts in [1] give

$$|W| = \left(\frac{27}{2} + o(1)\right) S.$$

The exact star-forest deficit theorem gives

$$\#\{w \in W : \deg(w) \neq 1\} = o(S).$$

A degree-one $w \in W$ is incident to a unique factorization edge $c = uw$ with $u \in B_0$. Since w is composite and $w \notin B_0$, this c belongs to the residual core C. Distinct such vertices give distinct elements, because every chosen edge has at most one endpoint in W. This produces C_0 and, by (4), proves (32).

Except for $o(S)$ elements, W is in bijection with $\mathcal{D}_n$. Indeed, a nondiagonal element $pq \in B_2$ with $p < q \le Y$ maps to (p, q), while an element $qr \in B_3$ maps to (r, q). Conversely, every pair in $\mathcal{D}_n$ with $q \le Y$ comes from B_2, and every pair with $Y < q \le n^{2/5}$ comes from B_3. The discarded sets are negligible: there are $O(M)$ squares, only $o(S)$ semiprimes in B_0, and

$$\sum_{q>n^{2/5}} \pi(n/q^2) \ll n \sum_{m>n^{2/5}} m^{-2} \ll n^{3/5} = o(S).$$

This proves (33) and (34).

Finally, pairs in $\mathcal{D}_n$ with $p \le n^{1/3-\varepsilon}$ are at most

$$\sum_{m \le n^{1/3-\varepsilon}} \sqrt{n/m} = O(n^{2/3-\varepsilon/2}) = o(S),$$

and pairs with $q \ge n^{1/3+\varepsilon}$ are at most

$$n \sum_{m \ge n^{1/3+\varepsilon}} m^{-2} = O(n^{2/3-\varepsilon}) = o(S).$$

The pairs with $q \le n^{1/3-\varepsilon}$ are even fewer. Together with (33), this proves (35). □

5. A locally private large-prime lemma

This section is the main new inverse step. It is a graph-valued version of the private-factor argument, applied one scale lower to the weights $w_{p,q}$.

5.1. **A small multiplicative basis at scale X.** For a parameter $X \to \infty$, use the same multiplicative basis construction at scale X. It gives a set

$$\mathcal{B}(X) = \mathcal{B}_0(X) \cup \mathcal{B}_1(X) \cup \mathcal{T}(X)$$

with the following properties:

(B1) every $d \le X$ has a chosen factorization

$$d = ab, \qquad a \in \mathcal{B}_0(X) = [1, X^{3/5}], \qquad b \in \mathcal{B}(X); \tag{36}$$

(B2) $\mathcal{B}_1(X)$ is the set of primes in $(X^{3/5}, X]$;

(B3)

$$|\mathcal{T}(X)| = O\left(\frac{X^{2/3}}{(\log X)^2}\right). \tag{37}$$

The set $\mathcal{T}(X)$ consists of the two secondary semiprime classes, with irrelevant overlaps absorbed into the O-term.

5.2. **The graph lemma.** Let $G = (\mathcal{V}, \mathcal{E})$ be a finite simple graph. For every edge $e = uv$, suppose we have an integer $d_e \le X$ and, for each incidence $v \in e$, an integer $L(v,e)$ divisible by d_e. Assume the cross condition

$$d_e \nmid L(u,f)L(v,g) \tag{38}$$

whenever $f \ne e$ is incident to u and $g \ne e$ is incident to v.

A divisor $x \mid d_e$ is called *locally private for e* if

$$x \nmid L(u,f) \quad (f \ne e,\ f \ni u), \qquad x \nmid L(v,g) \quad (g \ne e,\ g \ni v). \tag{39}$$

Lemma 5.1 (Locally private large prime)**.** *Under* (38)*, all but*

$$O\left(|\mathcal{V}|\left[X^{3/5}+\frac{X^{2/3}}{(\log X)^2}\right]\right) \tag{40}$$

edges e admit a locally private prime divisor

$$r_e > X^{3/5}. \tag{41}$$

The edges assigned the same prime r form a matching. Moreover,

$$d_e = r_e\lambda_e, \qquad \lambda_e < X^{2/5}. \tag{42}$$

Proof. Choose (36) for every d_e, say $d_e = a_e b_e$ with $a_e \in \mathcal{B}_0(X)$. For an endpoint $u \in e$, call a factor $x \in \{a_e, b_e\}$ *visible at u* if $x \mid L(u, f)$ for some other edge f incident to u.

The cross condition forbids a_e from being visible at one endpoint and b_e at the other; it also forbids the swapped arrangement. If neither factor is locally private, then each is visible at at least one endpoint. The two forbidden crossed arrangements force both factors to be visible at the same endpoint and invisible at the other endpoint. Charge e to the pair consisting of that other endpoint and the small factor a_e. This charging is injective: if two edges incident to the same vertex were charged to the same a, then each would make a visible for the other. Thus at most $|\mathcal{V}|\,|\mathcal{B}_0(X)|$ edges have no locally private chosen factor.

For every remaining edge, choose one locally private factor x_e. Edges with the same chosen factor form a matching, because two adjacent such edges would make that factor visible. Therefore the number of edges assigned a factor in $\mathcal{B}_0(X) \cup \mathcal{T}(X)$ is at most

$$\frac{|\mathcal{V}|}{2}\left(|\mathcal{B}_0(X)| + |\mathcal{T}(X)|\right).$$

All other chosen factors lie in $\mathcal{B}_1(X)$ and are primes exceeding $X^{3/5}$. This proves (40) and the matching statement. Finally $d_e \le X$ and $r_e > X^{3/5}$ imply $\lambda_e = d_e/r_e < X^{2/5}$. □

6. From weighted slots to a linear weighted triple core

6.1. **A compact central graph.** Fix a compact Jordan-measurable $K \Subset \mathcal{D}$. Let $G_n(K)$ be the graph whose edges are the pairs $(p, q) \in E_n$ from Theorem 4.1 satisfying

$$(p/Y, q/Y) \in K.$$

For fixed K, the prime number theorem gives

$$|E(G_n(K))| = (9\,\mathrm{area}(K) + o(1))S. \tag{43}$$

Its vertex set has size $O_K(M)$. If $e = pq$, write

$$c_e = pq\, w_e. \tag{44}$$

Since K is bounded away from the coordinate axes, there is a constant C_K such that

$$w_e \le C_K Y \tag{45}$$

for every edge of $G_n(K)$.

For an incidence $p \in e$, set

$$L(p,e) = \frac{c_e}{p}. \tag{46}$$

Then $w_e \mid L(p,e)$. If $e = pq$, $f \neq e$ is incident to p, and $g \neq e$ is incident to q, strong 2-primitivity gives

$$w_e \nmid L(p,f)L(q,g), \tag{47}$$

because otherwise

$$c_e = pqw_e \mid pL(p,f)\, qL(q,g) = c_f c_g.$$

Thus Theorem 5.1 applies with $X = C_K Y$.

After deleting $o_K(S)$ edges, we obtain for every remaining edge $e = pq$ a factorization

$$c_e = pq\, r_e \lambda_e, \qquad r_e \text{ prime}, \qquad r_e > (C_K Y)^{3/5}, \qquad 1 \le \lambda_e < (C_K Y)^{2/5}, \tag{48}$$

where r_e is locally private in the sense

$$r_e \nmid \frac{c_f}{p} \quad (f \neq e,\ f \ni p), \qquad r_e \nmid \frac{c_g}{q} \quad (g \neq e,\ g \ni q). \tag{49}$$

The error is indeed $o_K(S)$, because

$$|V(G_n(K))|(C_K Y)^{3/5} = O_K\left(\frac{Y^{8/5}}{\log n}\right) = o(S),$$

$$|V(G_n(K))|\frac{(C_K Y)^{2/3}}{(\log Y)^2} = O_K\left(\frac{Y^{5/3}}{(\log n)^3}\right) = o(S).$$

Edges for which $r_e \in \{p,q\}$ are also negligible: for each prime r, the edges assigned r form a matching, while every such edge is incident to r, so there is at most one. Delete these $O_K(M) = o(S)$ edges.

6.2. **The only remaining nonlinearity: reciprocal switches.** Associate to each remaining edge $e = pq$ the triple

$$T_e = \{p, q, r_e\}. \tag{50}$$

Lemma 6.1 (Reciprocal-switch lemma)**.** *Suppose e and f are distinct base edges. If $|T_e \cap T_f| \ge 2$, then the base edges are disjoint and*

$$r_e \in f, \qquad r_f \in e. \tag{51}$$

Proof. If e and f share a base endpoint, a second common point would mean that $r_e = r_f$, or that r_e is the other endpoint of f, or that r_f is the other endpoint of e. Each possibility contradicts (49). If the base edges are disjoint, two common points can only arise from the two reciprocal incidences in (51). □

We now show that reciprocal switches are negligible. Orient a reciprocal pair as (e, f). Write $e = uv$, with $r_f = u$, so that v is the *petal* of the oriented pair. Record

$$(v, \lambda_e, \lambda_f). \tag{52}$$

Lemma 6.2 (Switch count). *The map from oriented reciprocal pairs to the records* (52) *is injective. Consequently the number of oriented reciprocal pairs is at most*

$$|V(G_n(K))| \, (C_K Y)^{4/5} = o_K(S). \tag{53}$$

Proof. Write an oriented reciprocal pair explicitly as

$$e = \{u, v\}, \qquad f = \{r_e, w\}, \qquad r_f = u,$$

where v is the recorded petal. Once e and the choice of petal v are fixed, any reciprocal partner must be an edge through r_e whose assigned locally private prime is u. Two such partners would be adjacent and would have the same assigned prime u, contradicting the matching conclusion of Theorem 5.1. Thus the partner is unique.

Now suppose two distinct oriented reciprocal pairs (e, f) and (e', f') have the same record (v, a, b). By the preceding uniqueness, $e' \neq e$. Write

$$e = \{u, v\}, \quad r_e = R, \qquad e' = \{u', v\}, \quad r_{e'} = R'.$$

The reciprocal partner f has base edge $\{R, w\}$ and assigned prime $r_f = u$. Consequently c_f contains both R and u, while $c_{e'}$ contains v and the factor $\lambda_{e'} = a$. Since

$$c_e = uvR\lambda_e, \qquad \lambda_e = a \mid ba = \lambda_f \lambda_{e'},$$

we obtain $c_e \mid c_f c_{e'}$. The two factors on the right are distinct from the target c_e, so this contradicts strong 2-primitivity. This proves injectivity.

Since both multipliers lie in $[1, (C_K Y)^{2/5})$, the number of records is bounded by (53). Finally

$$\frac{M\,Y^{4/5}}{S} \asymp \frac{\log n}{Y^{1/5}} \longrightarrow 0.$$

□

Deleting one edge from every reciprocal pair removes $o_K(S)$ edges and, by Theorem 6.1, leaves a linear triple family.

Proposition 6.3 (Linear weighted core on a compact region). *For every fixed compact Jordan-measurable $K \Subset \mathcal{D}$, all but $o_K(S)$ weighted slots over K have a representation*

$$c_e = \lambda_e \prod_{p \in T_e} p, \tag{54}$$

where the T_e form a linear 3-uniform hypergraph and

$$1 \le \lambda_e < (C_K Y)^{2/5}.$$

The number of retained triples is

$$(9\,\mathrm{area}(K) + o(1))S.$$

7. Weighted near-extremizer classification

We can now exhaust the full feasible domain. Choose compact $K_j \Subset \mathcal{D}$ with

$$\mathrm{area}(K_j) > \frac{3}{2} - \frac{1}{j}.$$

Apply Theorem 6.3 and let $j = j(n) \to \infty$ sufficiently slowly. Together with Theorem 4.1, this gives a linear triple family of size $(27/2 - o(1))S$ representing all but $o(S)$ residual composites.

Theorem 7.1 (Small-hub weighted linear-triple classification)**.** *Assume* (1) *and the normalized stability theorem* (3)–(4)*. Let $A \subseteq [1, n]$ be strongly 2-primitive and satisfy* (2)*. After private-prime contractions, write*

$$\widetilde{A} = (\mathbb{P} \cap [1, n] \setminus V) \cup C.$$

Then there exist

- *a subfamily $C' \subseteq C$ with $|C \setminus C'| = o(S)$;*
- *a linear family $\mathcal{H}_n$ of triples of distinct primes;*
- *a bijection $E \mapsto c_E$ from $\mathcal{H}_n$ to C';*
- *positive integer multipliers λ_E;*

such that

$$c_E = \lambda_E \prod_{p \in E} p, \tag{55}$$

$$|\mathcal{H}_n| = \left(\frac{27}{2} - o(1)\right) S, \tag{56}$$

$$\lambda_E \prod_{p \in E} p \le n. \tag{57}$$

Moreover, for every fixed $\varepsilon > 0$, all but $o(S)$ edges satisfy

$$n^{1/3-\varepsilon} < p < n^{1/3+\varepsilon} \quad (p \in E), \qquad \lambda_E \le n^{\varepsilon}. \tag{58}$$

Equivalently, after deleting a further $o(S)$ edges, there is a sequence $\eta_n \to 0$ such that uniformly

$$n^{1/3-\eta_n} \le p \le n^{1/3+\eta_n} \quad (p \in E), \qquad \lambda_E \le n^{\eta_n}. \tag{59}$$

Consequently all prime divisors of all multipliers belong to a common hub $\mathcal{Q}_n$ with

$$|\mathcal{Q}_n| = n^{o(1)} = o(M), \qquad \mathcal{Q}_n \cap V(\mathcal{H}_n) = \varnothing. \tag{60}$$

Proof. The exhaustion argument preceding the theorem gives (55)–(56); the product bound is inherited from $c_E \le n$.

Sort an edge as $E = \{p < q < r\}$. Linearity makes the map $E \mapsto (p, q)$ injective, and $pqr \le n$ with $r > q$ implies $(p, q) \in \mathcal{D}_n$. By (30) and (56), the image misses only $o(S)$ feasible pairs. The pair-count estimates in the proof of Theorem 4.1 therefore put p and q in $n^{1/3+o(1)}$ for all but $o(S)$ edges. Since

$$q < r \le \frac{n}{pq},$$

the same is true of r. This proves the prime part of (58).

If all three vertex primes are at least $n^{1/3-\varepsilon/3}$, then

$$\lambda_E \le \frac{n}{pqr} \le n^{\varepsilon}.$$

This proves the multiplier part. A diagonal choice gives (59). Every prime divisor of a multiplier is at most n^{η_n}, whereas every hypergraph vertex is at least $n^{1/3-\eta_n}$; these ranges are disjoint for large n. Finally

$$|\mathcal{Q}_n| \le \pi(n^{\eta_n}) = n^{o(1)} = o(M),$$

which proves (60). ☐

7.1. **A converse.**

Proposition 7.2 (Weighted linear triples are sufficient)**.** *Let $\mathcal{H}$ be a linear family of triples of distinct primes. Let λ_E be positive integers whose prime divisors are disjoint from $V(\mathcal{H})$, and assume*

$$\lambda_E \prod_{p\in E} p \le n \qquad (E \in \mathcal{H}).$$

Put

$$A = \big(\mathbb{P} \cap [1, n] \setminus (V(\mathcal{H}) \cup H)\big) \cup \left\{ \lambda_E \prod_{p\in E} p : E \in \mathcal{H} \right\}, \tag{61}$$

where H is the set of prime divisors of the multipliers. Then A is strongly 2-primitive.

Proof. A retained singleton prime divides no composite. For a target edge E, any other hyperedge contains at most one vertex prime of E. Thus two other composites contain at most two of the three distinct vertex primes of E. Their multipliers use only primes outside $V(\mathcal{H})$ and cannot supply the missing vertex prime. The same argument applies when the two other elements coincide. ☐

Combining Theorems 7.1 and 7.2 gives the following normal-form statement at $o(S)$ resolution. Every normalized near-extremizer reduces to a near-saturated linear prime-triple system decorated by a subpolynomial common-hub weight, up to $o(S)$ exceptional residual elements; conversely, every weighted linear-triple system satisfying the product bound is strongly 2-primitive. What is not asserted is uniqueness of the near-saturated linear packing.

8. Consequences and interpretation

8.1. **What is now disproved.** The reflection construction proves that the statement

$$\#\{c \in C : c \text{ is not a squarefree product of three primes}\} = o(S)$$

is false. It fails in two independent ways:

- taking $m = 2$ creates a positive-density support-four layer $2pqr$;
- taking $m = 4$ creates a positive-density repeated-exponent layer 2^2pqr.

Both examples are reduced: every prime used by the residual core has degree at least two.

8.2. **What replaces squarefree-triple stability.** Theorem 7.1 gives the sharper replacement:

> After private-prime normalization and $o(S)$ deletions, every residual element is a small-hub multiplier times the product of a linear prime triple; the triple primes lie at scale $n^{1/3+o(1)}$, while the multiplier is $n^{o(1)}$.

Thus the weights are genuine moduli, not errors. Reflection completions show that positive-density fixed multiplier layers occur, so one should not expect a unique unweighted canonical construction.

8.3. **What remains open after this normal form.** The normal form does not classify the space of near-saturated linear triple packings themselves. Even in one fixed cell, different one-factorizations or colour permutations change $\Theta(S)$ triples. Nor does it classify the possible limiting distribution of the multipliers λ_E subject to the product slack. A natural next problem is therefore a transport problem: classify measurable completion laws and multiplier profiles that saturate the feasible-pair area $3/2$ while preserving pairwise linearity.

9. Independent verification checklist

The most delicate points of the draft are isolated in short statements that can be checked independently.

1. In the reflection construction, verify the interval arithmetic (15)–(16) and the three-role separation in Theorem 3.1.
2. Verify the capacity inequality (20), especially that compactness inside $\mathcal{D}$ forces $x \le 1 - \eta$.
3. Verify that a shared numerical pair has a unique role, and hence fixes the cell as in Theorem 3.2.

4. In Theorem 5.1, check the four possible visibility patterns of the two chosen factors. The only non-private admissible patterns put both factors at the same endpoint, allowing the injection into $\mathcal{V} \times \mathcal{B}_0(X)$.
5. In Theorem 6.2, check that two reciprocal switches with the same petal and the same ordered multiplier pair force a forbidden divisibility.
6. In the reflection diagonal, the compactness margin $1-(1-\eta_j)^{3/2}$ and the mesh width h_j both shrink. Choose the diagonal slowly enough that the prime-number-theorem error is smaller than the capacity margin and that $h_j M \to \infty$, so all used degrees tend to infinity.
7. Check the diagonal exhaustion $K_{j(n)} \nearrow \mathcal{D}$: the constants C_{K_j} may grow, so $j(n)$ must increase slowly enough that $C_{K_j} Y \leq n$ and the three errors

$$|V|X^{3/5}, \qquad |V|X^{2/3}/(\log X)^2, \qquad |V|X^{4/5}$$

remain $o(S)$.

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