A LOGARITHMIC-SQUARE CONSTRUCTION FOR SQUAREFREE PAIRWISE SUMS

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Abstract. We prove that there exists an infinite set $A \subset \mathbb{N}$ such that every sum $a + a'$ with $a, a' \in A$ is squarefree and

$$|A \cap [1, x]| \gg (\log x)^2.$$

Consequently, if $A = \{a_1 < a_2 < \cdots\}$, then $a_j \le \exp(C\sqrt{j})$ for some absolute constant C. The argument is based on a finite extension principle. Given a finite set $F \subset [1, N/2]$ with squarefree pairwise sums and $|F| = O((\log N)^2)$, the primes below a small multiple of $\log N$ are handled by placing all new elements in one Chinese-remainder class, while the remaining primes are handled by Konyagin's quadratic Brun sieve. Iterating the extension principle on dyadic scales gives the infinite set.

1. Introduction

Let SF denote the set of positive squarefree integers. We say that a set $A \subset \mathbb{N}$ has *squarefree pairwise sums* if

$$a + a' \in \mathsf{SF} \qquad (a, a' \in A),$$

where the case $a = a'$ is included. Erdős observed that a greedy argument produces an infinite set with this property whose elements grow exponentially, and asked how slowly such a sequence can grow [1].

For the finite problem, let ES_N be the largest cardinality of a subset of $[N] := \{1, \dots, N\}$ with squarefree pairwise sums. Konyagin proved

$$(\log N)^2 \log\log N \ll \mathrm{ES}_N \ll N^{11/15} \exp\Bigg(O\Bigg(\frac{\log N}{\sqrt{\log\log N}}\Bigg)\Bigg)$$

for large N [2]. The upper bound implies that every infinite sequence $A = \{a_1 < a_2 < \cdots\}$ with squarefree pairwise sums obeys $a_j \ge j^{15/11 - o(1)}$. Van Doorn and Tao recorded an infinite construction with

$$\log a_j \ll \frac{j}{\log j}$$

and noted that a suitable infinite version of Konyagin's finite construction should plausibly yield the stronger scale $\log a_j \ll \sqrt{j/\log j}$ [3].

The purpose of this note is to obtain an intermediate infinite bound by a comparatively direct extension argument.

Theorem 1.1. *There exists an infinite set $A \subset \mathbb{N}$ with squarefree pairwise sums such that*

$$|A \cap [1, x]| \gg (\log x)^2$$

for all sufficiently large x. In particular, writing $A = \{a_1 < a_2 < \cdots\}$, one has

$$a_j \le \exp(C\sqrt{j})$$

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for some absolute constant C and all $j \geq 1$.

The main point is that every sufficiently small finite good set can be extended at a prescribed much larger scale. For primes $p < \lambda \log N$, all new elements are placed in one residue class modulo p^2 that is compatible with the old set. The combined modulus is only $N^{2\lambda+o(1)}$, so the resulting progression still contains a power of N elements. Konyagin's quadratic Brun sieve then supplies many elements of that progression which avoid the congruence obstructions from all primes $p \geq \lambda \log N$.

2. Local restrictions

Call a finite set $F \subset \mathbb{N}$ *good* if $F + F \subset \mathsf{SF}$.

Lemma 2.1. *Let F be good.*

(1) *Every element of F is odd, and all elements of F lie in the same residue class modulo 4.*
(2) *For every odd prime p,*

$$0 \notin F \pmod{p^2}, \qquad (F \bmod p^2) \cap (-F \bmod p^2) = \varnothing.$$

Consequently,

$$|F \bmod p^2| \leq \frac{p^2-1}{2}.$$

Proof. For $f \in F$, the integer $2f = f + f$ is squarefree, hence $4 \nmid 2f$ and f is odd. If $f, g \in F$ represented different odd residue classes modulo 4, then $4 \mid f + g$, a contradiction.

For an odd prime p, the congruence $f \equiv 0 \pmod{p^2}$ would imply $p^2 \mid 2f$, and the congruence $f \equiv -g \pmod{p^2}$ would imply $p^2 \mid f + g$. Both are impossible. The nonzero residue classes modulo p^2 split into $(p^2-1)/2$ pairs $\{r, -r\}$, and F occupies at most one member of each pair. □

3. A uniform quadratic Brun-sieve consequence

We isolate the sieve input in the form needed below. For $y \geq 2$, put

$$P(y) := \prod_{p<y} p.$$

If $D \subset [N]$, $d \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, write

$$D_{d,\ell} := \{n \in D : n \equiv \ell \pmod{d^2}\}.$$

For each prime p in a set of sieving primes, let $B_p \subset \mathbb{Z}/p^2\mathbb{Z}$, let $\mathcal{B} = (B_p)_p$, and define

$$R(D, \mathcal{B}) := \{n \in D : n \bmod p^2 \notin B_p \text{ for every sieving prime } p\}.$$

Lemma 3.1 (A consequence of Konyagin's sieve)**.** *There is an absolute N_0 such that the following holds for every $N \geq N_0$. Set*

$$z_1 = \left\lfloor 10^{-3} \log N \right\rfloor, \qquad z_3 = \left\lfloor \frac{N^{1/10}}{2} \right\rfloor.$$

Suppose that $D \subset [N]$ satisfies

$$|D| > N^{19/20} \tag{1}$$

and

$$\left| |D_{d,\ell}| - \frac{|D|}{d^2} \right| \leq N^{1/100} \tag{2}$$

for every $\ell \in \mathbb{Z}$ and every d such that

$$d \le N^{3/20}, \qquad d \mid P(z_3), \qquad (d, P(z_1)) = 1.$$

For every prime $z_1 \le p \le \sqrt{N}$, let $B_p \subset \mathbb{Z}/p^2\mathbb{Z}$ satisfy

$$|B_p| \le 10^{-8}(\log N)^2. \tag{3}$$

Then

$$|R(D, \mathcal{B})| > N^{9/10}.$$

Proof. We apply Lemmas 5–7 of Konyagin [2, Section 4]. Besides z_1 above, choose

$$z_2 = \lfloor (\log N)^2 \rfloor, \quad \rho = \left\lceil 10^{-8}(\log N)^2 \right\rceil,$$
$$\alpha_1 = \frac{1}{100}, \quad \alpha_2 = \frac{1}{20}, \quad \alpha_3 = \frac{1}{10},$$
$$\delta_1 = \frac{1}{1000}, \quad \delta_2 = \frac{1}{20}, \quad \delta_3 = \frac{1}{100}, \quad \varepsilon = \frac{1}{50}.$$

We verify the numerical hypotheses, using the numbering in [2, Section 4]. First,

$$\alpha_1 \log\left(\frac{2\delta_1}{\alpha_1}\right) + \delta_1 = \frac{1}{100}\log\frac{1}{5} + \frac{1}{1000} < 0,$$

which is condition (4.1). Since $z_1 = (10^{-3} + o(1)) \log N$, condition (4.2) follows from

$$10^{-8} < \frac{10^{-6}}{2}$$

and

$$\frac{\rho}{z_1 \log z_1} = \left(10^{-5} + o(1)\right) \frac{\log N}{\log\log N} < \frac{1}{2000}\frac{\log N}{\log\log N}.$$

Condition (4.3) holds because

$$\alpha_1 \frac{\log z_2}{\log\log N} = \frac{1}{50} + o(1) < \frac{1}{20}.$$

Moreover, $z_2 \ge \max(\rho, \log N)$ and $z_3 \le N^{\alpha_3}/2$ for large N, which are (4.21) and (4.22).

The distribution range in Konyagin's condition (4.23) is $d \le N^{\alpha_2+\alpha_3} = N^{3/20}$, exactly as in (2). The principal exponent condition (4.24) is satisfied because

$$\alpha_1 \frac{\log(z_1^2 z_2)}{\log\log N} + 2\alpha_3 + \delta_2 + \delta_3 = \frac{4}{100} + \frac{2}{10} + \frac{1}{20} + \frac{1}{100} + o(1) = 0.30 + o(1) < 1 - \varepsilon.$$

Finally, Konyagin's conditions (4.30) and (4.31) reduce to

$$z_1^2 \le z_3 N^{-7/100}, \qquad z_1^2\sqrt{N} \le N^{93/100},$$

and both hold for all sufficiently large N.

Thus Konyagin's Lemma 7 gives

$$|R(D, \mathcal{B})| = (1 + o(1))\gamma_0 |D|, \qquad \gamma_0 := \prod_{z_1 \le p < z_2} \left(1 - \frac{|B_p|}{p^2}\right). \tag{4}$$

The use of (4) here is uniform over the positions of the residue sets B_p and over all D satisfying (1)–(2). Indeed, the error estimates in the proof of Konyagin's Lemmas 5–7, in particular his equations (4.17), (4.29) and (4.33), use only the displayed cardinality and discrepancy bounds, not the locations of the forbidden residue classes.

Konyagin's equation (4.18) yields

$$-\log\gamma_0 \le 0.9\delta_1\frac{\log N}{\log\log N},$$

so $\gamma_0 = N^{-o(1)}$. Combining this with $|D| > N^{19/20}$ in (4) gives $|R(D,\mathcal{B})| > N^{9/10}$ for sufficiently large N. □

4. The finite extension principle

Proposition 4.1. *There are absolute constants $c > 0$ and N_1 such that the following holds. For every $N \ge N_1$ and every good set $F \subset [1, N/2]$ with*

$$|F| \le c(\log N)^2,$$

there is a good set $\widetilde{F} \subset [N]$ satisfying

$$F \subset \widetilde{F}, \qquad |\widetilde{F}| = \left\lfloor c(\log N)^2 \right\rfloor, \qquad \widetilde{F} \setminus F \subset (N/2, N].$$

One may take $c = 10^{-9}$.

Proof. Let

$$z = \left\lfloor 10^{-3}\log N \right\rfloor, \qquad L = \left\lfloor 10^{-9}(\log N)^2 \right\rfloor.$$

By Lemma 2.1, all elements of a nonempty F lie in one class $\eta \in \{1,3\}$ modulo 4; if F is empty, set $\eta = 1$.

For each odd prime $p < z$, define

$$\Omega_p := \{0\} \cup \{-f \bmod p^2 : f \in F\}.$$

Lemma 2.1 gives

$$|\Omega_p| \le 1 + \frac{p^2-1}{2} = \frac{p^2+1}{2} < p^2.$$

Choose a residue $r_p \pmod{p^2}$ outside Ω_p. In particular,

$$r_p \not\equiv 0 \pmod{p^2}, \qquad r_p \not\equiv -f \pmod{p^2} \quad (f \in F). \tag{5}$$

Put

$$W := 4\prod_{\substack{3\le p<z\\ p \text{ prime}}} p^2.$$

By the Chinese remainder theorem, there is a residue $r \pmod W$ such that

$$r \equiv \eta \pmod 4, \qquad r \equiv r_p \pmod{p^2} \quad (3 \le p < z).$$

Let

$$D := \{n \in \mathbb{N} : N/2 < n \le N,\ n \equiv r \pmod W\}.$$

We first check the hypotheses of Lemma 3.1. If $\vartheta(y) = \sum_{p<y}\log p$, the prime number theorem gives

$$\log W = 2\vartheta(z) + O(1) = (2\cdot 10^{-3} + o(1))\log N.$$

Consequently,

$$|D| = \frac{N}{2W} + O(1) > N^{19/20}$$

for sufficiently large N.

Suppose that $d \mid P(z_3)$ and $(d, P(z)) = 1$, where z_3 is as in Lemma 3.1. Then $(d, W) = 1$. For every ℓ, the two congruences

$$n \equiv r \pmod W, \qquad n \equiv \ell \pmod{d^2}$$

combine into one class modulo Wd^2. Hence

$$|D_{d,\ell}| = \frac{N}{2Wd^2} + O(1), \qquad \frac{|D|}{d^2} = \frac{N}{2Wd^2} + O\left(\frac{1}{d^2}\right),$$

and therefore

$$\left| |D_{d,\ell}| - \frac{|D|}{d^2} \right| = O(1) < N^{1/100}.$$

Thus D satisfies the size and balance assumptions of Lemma 3.1.

We now enlarge F greedily inside D. Suppose that a good set

$$E = F \cup \{a_1, \dots, a_s\}, \qquad a_i \in D,$$

has already been constructed and $|E| < L$. For every prime $z \le p \le \sqrt{N}$, set

$$B_p(E) := \{0\} \cup \{-e \bmod p^2 : e \in E\}.$$

Then

$$|B_p(E)| \le |E| + 1 \le L \le 10^{-8}(\log N)^2$$

for sufficiently large N. Lemma 3.1 gives

$$|R(D, \mathcal{B}(E))| > N^{9/10}.$$

Since $|E| = O((\log N)^2)$, we may choose

$$a \in R(D, \mathcal{B}(E)) \setminus E.$$

We claim that $E \cup \{a\}$ is good.

Let $e \in E$. For $p = 2$, all elements of $E \cup \{a\}$ are congruent to η modulo 4, so $4 \nmid a + e$. If $p < z$ is odd and $e \in F$, then (5) gives $a + e \not\equiv 0 \pmod{p^2}$. If $p < z$ is odd and $e \in D$, then

$$a + e \equiv 2r_p \not\equiv 0 \pmod{p^2}.$$

For $z \le p \le \sqrt{N}$, the definition of $R(D, \mathcal{B}(E))$ gives $a \not\equiv -e \pmod{p^2}$, hence $p^2 \nmid a + e$.

It remains to consider $p > \sqrt{N}$. If $p^2 \mid a + e$, then $0 < a + e \le 2N < 2p^2$, so $a + e = p^2$. The left-hand side is 2 modulo 4, whereas the square of the odd prime p is 1 modulo 4, a contradiction. Thus $a + e$ is squarefree for every $e \in E$.

The self-sum $2a$ is also squarefree. The prime 2 causes no obstruction because a is odd. For an odd prime $p < z$, we have $a \equiv r_p \not\equiv 0 \pmod{p^2}$; for $z \le p \le \sqrt{N}$, the residue 0 belongs to $B_p(E)$; and for $p > \sqrt{N}$, the inequality $a \le N < p^2$ rules out $p^2 \mid a$. Therefore $E \cup \{a\}$ is good.

Repeating this step until the cardinality reaches L proves the proposition. □

5. Iteration and proof of the main theorem

Proof of Theorem 1.1. Choose N_1 sufficiently large for Proposition 4.1, and set

$$N_{k+1} = 2N_k \qquad (k \ge 1).$$

Starting from the empty set, Proposition 4.1 gives a good set $F_1 \subset [N_1]$ with

$$|F_1| = \left\lfloor c(\log N_1)^2 \right\rfloor.$$

Suppose inductively that $F_k \subset [N_k]$ is good and

$$|F_k| = \left\lfloor c(\log N_k)^2 \right\rfloor.$$

We have $N_k = N_{k+1}/2$ and $|F_k| \le c(\log N_{k+1})^2$. Applying Proposition 4.1 with $N = N_{k+1}$ produces a good set $F_{k+1} \subset [N_{k+1}]$ such that

$$F_k \subset F_{k+1}, \qquad |F_{k+1}| = \left\lfloor c(\log N_{k+1})^2 \right\rfloor.$$

Define

$$A := \bigcup_{k\geq 1} F_k.$$

Any two elements of A lie together in some F_k, so A has squarefree pairwise sums.

If $N_k \leq x < N_{k+1} = 2N_k$, then $\log N_k \geq \log x - \log 2 \geq \frac{1}{2}\log x$ for sufficiently large x, and hence

$$|A \cap [1, x]| \geq |F_k| \geq \frac{c}{8}(\log x)^2.$$

This proves the asserted counting-function estimate. Inverting it gives $a_j \leq \exp(C\sqrt{j})$ for all sufficiently large j, and increasing C absorbs the finitely many remaining indices. □

6. Remarks

Remark 6.1 (Relation to Konyagin's finite lower bound). The construction above deliberately uses only one residue class modulo

$$4\prod_{p<z} p^2.$$

For this class to have polynomially many representatives below N, one must keep $z = O(\log N)$. Konyagin's sieve condition permits roughly

$$\rho \ll z \log z \, \frac{\log N}{\log\log N}$$

for the number of forbidden residues per prime. With $z \asymp \log N$, this naturally produces $\rho = O((\log N)^2)$. Recovering the additional $\log\log N$ present in Konyagin's finite theorem appears to require a more flexible small-prime construction than a single Chinese-remainder class.

Remark 6.2 (The remaining gap). The theorem gives

$$\log a_j \ll \sqrt{j},$$

whereas the finite lower construction of Konyagin suggests the stronger possible infinite scale

$$\log a_j \ll \sqrt{\frac{j}{\log j}}.$$

On the other hand, the best known general lower bound remains $a_j \geq j^{15/11-o(1)}$ through Konyagin's finite upper bound.

Declaration of authorship and AI assistance

The author-originated inputs in the recorded development of this manuscript were the choice of Erdős's problem, the direction to pursue an explicit infinite construction seriously, and the decision to formulate and circulate the resulting theorem. The author, Xiyu Hu, retains sole responsibility for the mathematical claims and for the final form of the manuscript.

OpenAI's GPT-5.6 Pro was used substantially in preparing this draft. Its assistance included proposing the decomposition in which a Chinese-remainder restriction handles primes $p < \lambda \log N$ and Konyagin's quadratic Brun sieve handles the remaining primes; carrying out an initial audit of the numerical parameters in Lemma 3.1; locating and summarizing the cited literature; and drafting, reorganizing, and typesetting substantial portions of the manuscript. Thus, the central proof strategy and much of the initial wording were developed with generative-AI assistance rather than being claimed as unaided work of the author.

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